

# The overlapping of nonlinear resonances and the problem of quantum chaos

A. Ugulava, L. Chotorlishvili, K. Nickoladze.

*Tbilisi State University, Department of Physics,*

*Chavchavadze av. 3, 0128 Tbilisi, Georgia*

(February 8, 2008)

## Abstract

The motion of nonlinearly oscillating particle under the influence of a periodic sequence of short impulses has been investigated. Analysis of the Schrodinger equation for the universal Hamiltonian has been done. It is shown that the quantum criterion of overlapping of resonances is of form  $\lambda K \geq 1$ , where  $K$ -is the classical coefficient of stochasticity, and  $\lambda$  is the functional defined with the using of Mathieu functions. The area of maximal values of  $\lambda$  is determined. The idea about emerging of quantum chaos originated due to the adiabatic motion along the curves of Mathieu-characteristics at multiple passages through the points of branching is advanced.

PACS number: 87.22.Jb

## I. INTRODUCTION

The overlapping of nonlinear resonances is the criterion for the origin of dynamical stochasticity in classical Hamiltonian systems. At realization of the condition of this criterion it is possible to justify the transition from the dynamic Hamilton description to the statistical one and to study the behavior of the system with the help of statistical average. Such description is as full as possible in this range and successfully substitutes dynamic description which loses its sense due to strong local instability [1,2]. However in the quantum mechanics the introduction of a stochasticity is significantly difficult [3–8]. What can be considered as quantum analog of dynamic stochasticity, what is a criterion for passing to quantum chaos, how to quantize the system in classical limit corresponding to the dynamic stochasticity? There are only some problems of quantum chaos.

In the given work the attempt is made to investigate two aspects of a general problem of quantum chaos: the criterion for the overlapping of resonances on the basis of quantum mechanics and to study the singularities of wave functions in the area, in which the classic mechanics assumes the existence of dynamic stochasticity.

Let us assume, that nonlinearly oscillating particle (Fig. 1) is under the action of variable field

$$H(x, p) = H_o(x, p) + H_{NL}(x) + \varepsilon V(x, t), \tag{1}$$

where

$$H_o(x) = 1/2(p^2/m + \omega_o^2 m x^2),$$

$$H_{NL} = \gamma x^3 + \beta x^4 + \dots,$$

$$\varepsilon V(x, t) = -(e/m)x f(t),$$

$$f(t) = f_o \Sigma(t) \cos \omega t, \quad \varepsilon V(x, t) = \varepsilon V_o x \Sigma(t) \cos \omega t,$$

$$\varepsilon V_o = -(e/m)f_o,$$

$$\varepsilon \ll 1. \tag{2}$$

The Hamiltonian of such type for a long time is being investigated with the purpose of study of a dynamic stochasticity both in classical [1], and in quantum systems [3,5,6].

Here  $x$  and  $p$  are the coordinate and the impulse of the particle,  $\omega_o$  is the fundamental frequency,  $\gamma$  and  $\beta$  are the coefficients of the nonlinearity,  $m$  and  $e$  are the mass and the charge of the particle,  $f_o$  is the amplitude of the variable field,  $\Sigma(t)$  is the periodic sequence of rectangular electromagnetic impulses with the duration  $\tau$  and with the phase of recurring  $T$  (Fig.2, Fig.3). It is supposed, that  $1/\omega, 1/\omega_o \ll \tau \ll T$ .

The fundamental component of pumping field at frequency  $\omega = \omega_o$  is able to carry out the linear resonance and cause the increase of  $x$  until the nonlinear terms proportional to  $x^3$  and  $x^4$  become significant in the potential (i.e. up to a neighborhood of  $x_L$ , Fig. 1). From this moment the nonlinear terms will gradually begin to detune the linear resonance (at  $\omega = \omega_o$ ), that will reduce the resonance growth of  $x$ . Then the remaining harmonics of the pumping spectrum, concentrated in sum  $\Sigma(t)$ , will begin to play the role. Their role will be significant in reaching higher excitation ( $x > x_L$ ), if criteria of the overlapping of resonances is fulfilled.

## II. UNIVERSAL HAMILTONIAN. CLASSICAL CONSIDERATION

In this paragraph we shall review the well-known results obtained in the theory of a stochasticity for the nonlinearly oscillating classical systems. After passing in the Hamiltonian (1) to the variables action - angle with help of transformation  $x = \sqrt{2I/m\omega_o} \cos \theta$ ,  $p = -\sqrt{2I\omega_o m} \sin \theta$ , and averaging (1) with respect to the fast phase  $\theta$ , we obtain:

$$H = H_o^{NL} + \varepsilon V(I) \dots,$$

$$H_o^{NL} = H_o + H_{NL},$$

$$H_o = I\omega_o, H_{NL} = 3\pi\beta(I/m\omega_o)^2,$$

$$\varepsilon V(x, t) = \varepsilon/2 V(I) \cos \varphi \Sigma(t) \equiv \varepsilon V(I, \varphi, t), \quad V(I) = V_o \sqrt{2I/m\omega_o} \quad (3)$$

where  $\varphi = \theta - \omega t$  is the slow phase.

Let us notice, that  $H_{NL}$  we have united with  $H_o$  in unperturbed Hamiltonian  $H_o^{NL}$ . In the further nonlinear terms are not assumed small and the application for them of the perturbation theory is not possible. The relevant set of canonical equation looks like:

$$\dot{I} = -\varepsilon \frac{\partial V(I, \varphi, t)}{\partial \phi},$$

$$\dot{\varphi} = \omega(I) + \varepsilon \partial V(I, \varphi, t) / \partial I,$$

where

$$\omega(I) = \omega_o - \omega + \omega_{NL}(I), \quad \omega_{NL} = 6\pi\beta I / (m^2 \omega_o^2), \quad (4)$$

$$\varepsilon V(I, \varphi, t) = 1/2 \frac{\tau}{T} V(I) \cos \varphi \sum_{-1/\tau}^{1/\tau} \cos k\Omega t. \quad (5)$$

The phase  $\varphi$ , slow as compared to the  $\theta$  remains fast in the comparison with the velocity of the action  $I$  variation. The velocity of the variation of  $\varphi$  contains information about the nonlinear character of motion. In particular, the dependence of  $\dot{\varphi}$  from  $I$  means the presence of nonlinearity in the oscillating system. Suppose that for some values of the  $I_n$  the resonance between  $\omega(I)$  and some component from the polychromatic pumping spectrum (5) (i.e.  $\omega(I_n) \approx n\Omega$ ) is carried out. Then forming a slow phase  $\alpha_n \approx \varphi - n\Omega t$ , averaging expression (5) with respect to the fast  $\varphi$  and taking into account (5), we get:

$$\varepsilon V(I, \varphi, t) = 1/4 \frac{\tau}{T} V(I) \cos \alpha_n. \quad (6)$$

Substituting (6) in (4), we have:

$$\dot{I}_n = U(I) \sin \alpha_n, \quad (7)$$

$$\dot{\alpha}_n = \omega(I_n) - n\Omega + dU(I)/dI \cos \alpha_n,$$

where

$$U(I) = 1/4 \frac{\tau}{T} V(I). \quad (8)$$

Equations (7) describe the nonlinear resonance. As opposite to the linear resonance at which unbounded linear growth of an amplitude is valid (in our case action  $I$  or deviation  $x$ ), in the case of the nonlinear resonance (as it was already mentioned), there are so-called "phase oscillations", i.e. oscillation of the phase  $\alpha_n$  and the amplitude  $I_n$ .

Let us introduce the deviation of the action,  $\Delta I_n = I - I_n$ ,  $\Delta I \ll I_n$  from the resonance value. Then it is possible to demonstrate that the Hamiltonian

$$\tilde{H} = \omega'(\Delta I)^2/2 + U(I_n) \cos \alpha_n, \quad (9)$$

where  $\omega' = (d\omega/dI)_{I=I_n}$ , produces the set of equations of (7) type. Really, from the equilibrium conditions,  $\dot{\alpha} = \dot{I} = 0$ , one can obtain:

$$\omega(I_n) - n\Omega + \varepsilon/2 dU(I_n)/dI = 0, \quad (10)$$

$$\omega(I_n) = \omega(I_n) + \omega' \Delta I_n.$$

If the condition of moderate nonlinearity is just  $\varepsilon \ll \mu \ll 1/\varepsilon$ , where

$$\mu = I_n/\omega(I_n) \left( \frac{d\omega}{dI} \right)_{I=I_n}, \quad (11)$$

is the factor of nonlinearity, then with the help of (7)-(11), we get

$$\Delta \dot{I}_n = U(I_n) \sin \alpha_n, \quad (12)$$

It is possible to obtain the equation  $\dot{\alpha}_n = \omega' \Delta I_n$  for phase oscillations from the set (12)

$$\ddot{\alpha}_n - \omega_{ph}^2 \sin \alpha_n = 0, \quad (13)$$

where  $\omega_{ph} = \sqrt{\omega' U}$  is the frequency of phase oscillations.

Let us notice, that the Hamiltonian (9) is the Hamiltonian of the mathematical pendulum, where the role of the mass plays  $1/\omega'$  and the role of the potential energy -  $U(I_n) \cos \alpha_n$ . Taking into account that in the classical mechanics the problem of the pendulum can be solved exactly we reduced an initial problem to the solved one.

Variation of the action with the help of (9) and (12) can be presented in the form:

$$\Delta I_+ = \sqrt{(E+U)/\omega'} dn(\omega' \sqrt{(E+U)/\omega'} t; k), \quad E > U \quad (14)$$

(with the period equal to  $2K(k)$ ),

$$\Delta I_- = \sqrt{(E+U)/\omega'} cn(\omega' \sqrt{(E+U)/\omega'} t; 1/k), \quad E < U, \quad (15)$$

(with the period equal to  $4K(1/k)$ ), where  $E$  is the energy of the particle,  $cn$  and  $dn$  are Jacobian elliptic functions: the elliptic cosine and delta of amplitude;  $K(k)$  is the second order complete elliptic integral,  $k = \sqrt{2U/(E+U)}$  is the module of elliptic integrals.  $\Delta I_+$  and  $\Delta I_-$  are deviations of the action up and below the separatrix accordingly. For  $E = U$  (or  $k \rightarrow 1$ ) these two solutions are sewed together and take the form of an instanton

$$\Delta I_+ \rightarrow \Delta I_- \rightarrow \frac{\sqrt{2U/\omega'}}{c\hbar(\sqrt{2U\omega'}t)}. \quad (16)$$

Averaging the action deviation with respect to half period we get the following equations:

$$\begin{aligned} \overline{\Delta I_+} &= \sqrt{(E+U)/\omega'} \frac{1}{K(k)} \int_0^{4K(k)} dn(\tau, k) d\tau = \\ &= \frac{\pi \sqrt{E+U/\omega'}}{2K(k)}, \quad E > U; \end{aligned} \quad (17)$$

$$\begin{aligned} \overline{\Delta I_-} &= \sqrt{(E+U)/\omega'} \frac{1}{2K(1/k)} \int_{\alpha_0}^{2K(1/k)} cn(k\tau, 1/k) d\tau = \\ &= \frac{\pi \sqrt{U/2\omega'}}{K(1/k)}, \quad E < U; \end{aligned} \quad (18)$$

$$\alpha_o = \arccos(-E/U).$$

At  $E \approx U$

$$\overline{\Delta I_-} \approx \overline{\Delta I_+} \approx \overline{\Delta I} = \pi \sqrt{U/2} \frac{1}{\ln 4 \sqrt{4U/(U-E)}}. \quad (19)$$

Action variation dependence on the ratio  $E/U$  is presented on the Fig.4.

According to Fig.4, the magnitude of  $\Delta I_-$  sharply decreases with approaching to a separatrix.

If during the phase oscillation  $\Delta I_n$  takes enough major values (such as  $\omega' \Delta I_n \geq \Omega$ ), the resonance condition  $\omega(I_n) \approx n\Omega$  breakes, but other resonance condition is attuned:

$$\omega(I_n^o + \Delta I_n) \approx (n+1)\Omega. \quad (20)$$

Just at the jump to the other resonance condition there is an abruptness, which results in a stochastic wandering of spectral harmonics (6). It is the essence of overlapping of resonances, which serves as the criterion of the stochasticity emerging in the nonlinearly oscillating system. Expending  $\omega(I)$  into series with respect to  $\Delta I$ , and making an estimation,  $\Delta I \approx \sqrt{U/\omega'}$ , on the bases of (17) and (18), it is possible to present the condition (20) in the form  $\sqrt{\omega'U} \approx \Omega$ . Thus, in the case of overlapping of resonances the phase oscillations frequency -  $\omega_{ph}$ , coincides by an order of magnitude with the frequency distance between harmonics in the pumping spectrum. Usually criterion of the dynamic stochasticity, equivalent to the overlapping of resonances, is written down by introducing the stochasticity coefficient

$$K \approx \sqrt{\omega'U}/\Omega > 1. \quad (21)$$

In the range of statistical motion the nonlinear oscillating system is described with the help of distribution function  $\rho(t)$ , for which it is possible to obtain the diffusion equation [1]

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho(I, t)}{\partial I^2}, \quad (22)$$

where  $D = \frac{1}{2}U^2(I)T$  is a diffusion coefficient. Now from (22) with the help of  $\rho(I, t)$  it is easy to get  $\bar{I} = \bar{I}_0 + Dt$ , where average is understood as a statistical average. Diffusion growth of

the action reduces the growth of  $\sqrt{\langle x^2 \rangle}$  in the range of  $x > x_L$ . As an energy of the particle, located in a hole, is  $E_o = I\omega_o$ , then in the range of stochastic dynamics  $\overline{E(t)} = E_o + \omega_o\sqrt{Dt}$  and "heating" of the particle takes the place. The above-mentioned reasonings are proved by numerical calculations (Fig. 5).

The first numerical experiments for stochastic "heating" of a nonlinear oscillator were carried out long time ago, see [1].

The condition of resonances overlapping has visual interpretation on a phase plane.

The mathematical pendulum, in an association with initial conditions, can make two types of motion: oscillatory and rotary. They are separated by a separatrix on the phase diagram. The overlapping of resonances on the phase plane corresponds to a touch of separatrices (Fig. 6), if the width of the separatrix is estimated as  $\omega'\Delta I_n \approx \sqrt{\omega'U}$  (in frequency units).

The dynamic stochasticity usually originates in a narrow layer near to a separatrix [1,2]. Therefore, at quantum reviewing we shall be especially interested in quantum properties of the system near to the separatrix. In other words, we shall be interested in a wave function of isolated nonlinear resonance in the absence of overlapping. The analysis of these properties can explain essence of a quantum chaos.

In conclusion of this paragraph we shall remark, that the condition of overlapping of resonances depends on the action  $\Delta I_n$  as the solution of the equations generated by the universal Hamiltonian. Therefore, at quantum reviewing, for establishment the criterion of overlapping of resonances it is enough in (20) under  $\Delta I_n$  to understand the relevant magnitude obtained from the quantum equations generated by the Hamiltonian (9).

Except the quantum estimation for  $\Delta I$  we shall be interested also in quantum dynamics near to the separatrix ( $E \approx U$ ), as the basis of directed random motion in one-dimensional nonlinear systems.



### III. QUANTUM-MECHANICAL CONSIDERATION

The quantummechanical consideration of isolated nonlinear resonance, as well as overlapping of two resonances was presented in [3,5,6]. In [5] it was specified, that the Schrodinger equation for a nonlinear isolated resonance can be reduced to the Mathieu's equation, and if the condition of the resonances overlapping is fulfilled the correlations drop [6] (numerical computational methods). We in this paragraph are interested essentially in quantum-mechanical characteristics of the problem - wave function and energy spectrum. We shall investigate, how can at quantum case the unpredictability of hit of a system in any quantum state (analog of a stochastic stratum near a separatrix in classical mechanics).

The universal Hamiltonian (9) depends on the basic parameter of nonlinear oscillations,  $\omega'$ . At quantum reviewing corresponding to the universal Hamiltonian Schrodinger equation will be also depending on  $\omega'$ . So we came to the quantum-mechanical consideration of the nonlinear-oscillating system within the frame of approximation made in §1.

The Schrodinger equation relevant to the Hamiltonian (9), is:

$$\frac{d^2\Psi}{d\alpha^2} + \frac{2}{\chi}[E - U \cos \alpha]\Psi = 0, \quad (23)$$

where  $\chi = \omega'\hbar^2$ .

Let us clarify essence of the parameter  $\chi$ . The value  $\omega'\hbar$  is the quantum (the minimal portion) of the frequency shift stipulated by nonlinearity (i.e., the frequency quantum of nonlinearity). Hence, value  $\chi = \omega'\hbar^2$  is the energy quantum of nonlinearity.

The equation (23) is the Mathieu's equation which analysis we shall make below. For now we want to get quasi-classical wave functions relevant to an approximation  $\Lambda = U/\chi \gg 1$  of the Schrodinger equation. It is known, that quasi-classical wave function is:

$$\Psi(\alpha) = \frac{c}{\sqrt{\Delta I}} \exp\left(i/\hbar \int_0^\alpha \Delta I(\alpha) d\alpha\right), \quad (24)$$

where  $c$  is the normalizing constant and  $\Delta I$  can be find from the integral of the energy

$$\Delta I = \sqrt{2/\omega'(E + U \cos \alpha)}. \quad (25)$$

Substituting (24) in (23), after integration we get:

$$\Psi_+(\alpha) = 1/2 \frac{(E+U)^{1/4} \exp\left(2i\sqrt{(E+U)/\chi} E(\alpha/2, k)\right)}{\sqrt{K(k)} (E+U \cos \alpha)^{1/4}}, \quad (26)$$

$$E > U, 0 \leq \alpha \leq \pi; \quad \Psi_-(\alpha) = 1/\sqrt{2} \frac{(U/2)^{1/4}}{\sqrt{K(1/k)}} \times$$

$$\times \frac{\exp\left(i\sqrt{\frac{2}{U\chi}}[(E-U)F(\gamma, 1/k) + 2UE(\gamma, 1/k)]\right)}{(E+U \cos \alpha)^{1/4}}, \quad (27)$$

$$0 < E < U, \quad 0 \leq \alpha \leq \alpha_o;$$

$$\gamma = \arcsin \sqrt{\frac{U(1 - \cos \alpha)}{U + E}}, \alpha_o = \arccos(-E/U), \quad (28)$$

where  $F(\dots)$  is the elliptic integral of the first kind,  $E(\dots)$  is the elliptic integral of the second kind and  $K(\dots)$  is the first kind complete elliptic integral.

Let's note, that as in the considered case  $\Lambda \gg 1$  quasi-classical wave functions (26) and (27) oscillate fast with the variation of  $\alpha$ , having peaks at turning points  $\pm\alpha_o$ . It is a common property of wave functions in a quasi-classical approximation.

The wave functions corresponding to the separatrix can be obtained in the limit  $E \rightarrow U$ :

$$\begin{aligned} \Psi_-(\alpha) &= \Psi_+(\alpha) = \Psi_s(\alpha) = \\ &= 1/2\sqrt{2} \frac{1}{\ln 4 \sqrt{\frac{U}{|E-U|}}} \frac{\exp(i2\sqrt{2\Lambda} \sin \alpha/2)}{(1 + \cos \alpha)^{1/4}}. \end{aligned} \quad (29)$$

According to expression (29) for a wave function near the separatrix the frequency of fast oscillations practically does not vary. The turning points  $\alpha_o \langle \pi$  approach to  $\pm\pi$  and the peaks of fast oscillations become negligible low. This is connected with the logarithmically diverging factor in (29). With the help of (29) it is easy to find an equation for nodal points of separatrix wave function,  $\Psi_s(\alpha) = 0$

$$2\sqrt{2U/\chi} \sin \frac{\alpha_n}{2} = \pi/2 + 2\pi n, n = 1, 2, \dots \quad (30)$$

Differentiating the relation (24) it is possible to calculate the density of nodal points

$$\frac{dn}{d\alpha_n} = \frac{1}{2\pi} \sqrt{\frac{2U}{\chi}} \cos \frac{\alpha_n}{2}. \quad (31)$$

The high density of nodal points is provided by the major parameter of quasi-classical consideration  $(2U/\chi)^{1/2}$ , which is suppressed by the zeroes of the factor  $\cos \frac{\alpha_n}{2}$  in the points  $\alpha_n = \pm\pi$  (see Fig.7). Another important characteristic of the quantum state near to the separatrix is the density of energy levels. Bohr - Zommerfeld quantization condition looks like:

$$I = \oint \Delta I(\alpha) d\alpha = n\hbar. \quad (32)$$

Taking into account (25), we get:

$$\begin{aligned} I &= \sqrt{2/\omega'} \int \sqrt{E + U \cos \alpha} d\alpha = \\ &= \frac{2}{\sqrt{U\chi}} \left( (E - U)K(1/k) + 2UE(1/k) \right) = n. \end{aligned} \quad (33)$$

With the help of (33) for the density of energy levels it is possible to obtain:

$$\frac{dn}{dE} = \frac{1}{\sqrt{U\chi}} K(1/k) \quad (34)$$

In the limit  $E \rightarrow U$  for the energy level density,  $dn/dE$ , we get:

$$\left( \frac{dn}{dE} \right)_s = \frac{1}{\sqrt{\chi U}} \ln 4 \sqrt{\frac{2U}{|E - U|}}. \quad (35)$$

As it can be seen from (35) the level densities are logarithmic divergent near the separatrix.

Now let us try to evaluate quasi-classical condition of the overlapping of resonances. For this purpose it is necessary to calculate average values  $\Delta I$  for a halfcycle of motion both above and lower of the separatrices. Taking into account (25)-(27), we obtain:

$$\langle \Delta I \rangle^+ = \int_{-\pi}^{\pi} |\Psi_+(\alpha)|^2 \Delta I(\alpha) d\alpha = \pi/2 \sqrt{\frac{E+U}{\chi}} \frac{\hbar}{K(k)} \quad (36)$$

$$\langle \Delta I \rangle^- = \int_{-\alpha_o}^{\alpha_o} |\Psi_-(\alpha)|^2 \Delta I(\alpha) d\alpha = \sqrt{\frac{U}{2\chi}} \frac{\alpha_o \hbar}{K(1/k)} \quad (37)$$

and near the separatrixes,  $E \approx U$ :

$$\langle \Delta I \rangle^+ \approx \langle \Delta I \rangle^- \approx \langle \Delta I \rangle_S = \pi \sqrt{U/2\chi} \frac{\hbar}{\ln 4 \sqrt{\frac{2U}{|E-U|}}}, \quad (38)$$

where  $\langle \dots \rangle^\pm$  denotes averaging with the help of wave functions (26), (27) and (29). Let's note, that

$$\langle \Delta I \rangle^- / \hbar = (U/2\chi)^{1/2} (\alpha_o / K(1/k)) \approx \Lambda^{1/2} \gg 1$$

is in a good agreement with a quasi-classical condition of motion. Because of the coincidence of classical values  $\overline{\Delta I}_\pm$  with the quasi-classical ones  $\langle \Delta I \rangle^\pm$  it is natural that conditions of overlapping resonances will also coincide  $\omega' \overline{\Delta I}_- = \omega' \langle \Delta I \rangle^- = \Omega$ . Hence, one can conclude that under quasi-classical conditions  $\Lambda \gg 1$  and the condition of overlapping the resonances the stochastic "heating" of electron being under the conditions given in section 1 and the obtaining of high excitations can be carried out (see classical case, Fig.5). Quasi-classical expressions (36), (37) by the form coincide with the similar classical expressions (17) and (18). But as opposite to the classical formulas, in quasi-classical expressions under  $E$  should be understood  $E_n$  energy spectrum of quasi-classical levels. With the help (33) and (34) it is possible to determine number of the levels entrapped in a nonlinear resonance:

$$\Delta n = \frac{dn}{dE} \Delta E, \quad (39)$$

where

$$\Delta E = \hbar \omega' \langle \Delta I \rangle^-. \quad (40)$$

$\Delta n$  is the important characteristics of an isolated nonlinear resonance. Using (34), (37), (39) and (40) it is easy to show, that the number of levels entrapped in nonlinear resonance is

$\Delta n \simeq \frac{\alpha_o}{2}$ . According to (35), (38),(39) and (40) it is also easy to show, that near the separatrix there is  $\Delta n_s \simeq \frac{\pi}{2}$ . While according to (34) and (35) the density of levels increases logarithmically with approaching to the separatrix, the number of levels  $\Delta n$  entrapped in resonance is not increased. This is caused by the sharp fall of the action variation near  $U \simeq E$  (see Fig.4). Thus the number of entrapped levels in a nonlinear resonance is not great and remains the same when approaching to the separatrix. Major values  $\Delta n$ , as it was shown in [6] by means of numerical methods, can be reached in case of overlapping resonances.

Let's analyse the Schrodinger equation (23). We assume, that  $U$  and  $\chi$  are the values of one order and that is why the quasi-classical approximation can not be used.

In the limit case  $E \gg U$  it is possible to use limiting ( $U \rightarrow 0$ ) formulas for the Mathieu's functions [9]:

$$\Psi_0(\alpha) = ce_o(\alpha) = 1/\sqrt{2},$$

$$\Psi_n(\alpha) = ce_n(\alpha) = \cos n\alpha,$$

$$\Psi_n(\alpha) = se_n(\alpha) = \sin n\alpha, n = 1, 2, 3 \dots \quad (41)$$

These functions should satisfy equation

$$\frac{d^2\Psi}{d\alpha^2} + \frac{2E}{\chi}\Psi = 0 \quad (42)$$

which follows from (23) at  $U \rightarrow 0$ . The equation describes harmonic oscillations with the frequency  $\sqrt{2E/\chi}$ . To reduce it in the correspondence with the solution (41) it is necessary to require

$$\sqrt{\frac{2E}{\chi}} = n \quad \text{or} \quad E_n = 1/2\chi n^2, \quad (43)$$

where  $n = 1, 2, 3 \dots$ . The last relation leads to the energy spectrum quadratically depending on the quantum number.

It is possible to use the relations obtained with the help of an averaged universal Hamiltonian for calculation of  $\Delta I$  in the zero order with respect to the  $U/E$

$$\langle (\Delta I)^2 \rangle_n = \frac{2}{\omega'} E_n. \quad (44)$$

Then taking into account of (43) we get

$$\langle \Delta I \rangle_n \approx \sqrt{\langle (\Delta I)^2 \rangle_n} = n\hbar. \quad (45)$$

It is possible to obtain the condition of the overlapping of resonances for a frequency shift, caused by the variation of an action

$$\delta\omega \approx \omega' \langle \Delta I \rangle_n = \omega' n\hbar \geq \Omega. \quad (46)$$

Using (46) the condition of the resonances overlapping could be written as  $\delta\omega \approx \omega' n\hbar \geq \Omega$ . It is significantly difficult to fulfill this condition as compared with similar one of quasi-classical case, because it requires excessively small  $\Omega$ .

In the opposite limit case  $E \ll U$  we have condition of overlapping the resonances, which is also difficult to be fulfilled and by this reason we don't present a detailed analysis.

It is well known, that eigenvalues of the Mathieu's equation can be defined by means of the Ince-Strutt diagrams (Fig.9) constructed for the first time to studying a parametrical resonance [9]. As follows from these diagrams, each value  $U$  corresponds to set of eigenvalues  $E_n$  and periodical wave functions  $ce_n(\alpha), se_n(\alpha)$ . Curves in Fig.9 corresponding to the realized quantum states are known Mathieu-characteristics. Essential feature of Mathieu-characteristics is the presence of points of branching in the neighbourhood of a line  $U = E$ , corresponding to the classical separatrix. Moving along Mathieu-characteristics from left to right in points of branching being at the left of separatrix line disappears two-fold degeneration. So through the passing of separatrix line  $U = E$  and reaching the point of branching, to the right of a separatrix wave functions merge again but now -  $se_n$  and  $ce_{n-1}$ . The emergence of such picture (point of branching from two sides of the separatrix) at the passage to quantum consideration is a principal characteristic describing a quantum system

near classical separatrix. One can observe the appearance of unpredictability of occupied quantum levels with the help of branching points located on both sides of a separatrix. Let us suppose, that one of the system parameters, for example, the amplitude of variable field  $U$  varies adiabatically ( $U \rightarrow U + \epsilon \cos \kappa t, \epsilon \ll 1$ ,  $\kappa$ -frequency of slow motion). In common problems of a quantum mechanics is believed that the distance between levels in an energy distribution, depending on exterior parameters vary synchronously with adiabatically varying parameter, not changing in this case a quantum state. The situation is changed radically, if in a quantum mechanical problem of the definition of energy distribution and eigenfunctions is reduced to the analysis of diagrams of the Ince-Strutt type. At slow moving along the curves of Mathieu-characteristics due to the adiabatic change of the amplitude of variable field, after multiple passages through the branching points it is impossible to determine exactly in which Mathieu-characteristic (in which quantum state) the system can be found. One can consider this appearance as quantum analog of formation of a stochastic layer of motion in the area of classical separatrix.

Let us pass to the analysis of overlapping criterion in a quantum case. Taking into account the quantum virial theorem, connecting average kinetic energy  $T$  with the average potential energy  $U$

$$2 \langle T \rangle = \langle \alpha \frac{dU}{d\alpha} \rangle, \quad (47)$$

for variation of action  $\Delta I$  with the help of (44) we obtain:

$$\langle \Delta I \rangle \approx \sqrt{(\Delta I)^2} = \lambda(\Lambda) \sqrt{\omega' U}$$

where

$$\lambda(\Lambda) = \sqrt{- \int_0^\pi c e_n^2(\alpha, \Lambda) \alpha \sin \alpha d\alpha} \quad , \quad (48)$$

and  $\Lambda = U/\chi$ . The condition of overlapping of resonances takes the form:

$$\delta\omega = \omega' \langle \Delta I \rangle \approx \lambda(\Lambda) \sqrt{\omega' U} \geq \Omega \quad , \quad (49)$$

or

$$\lambda(\Lambda)K \geq 1 , \quad (50)$$

Comparing the quantum criterion (50) with the classical (21), one can conclude that in quantum case there is additional quantum factor  $\lambda(\Lambda)$ .

As can be seen from (50), conditions of resonances overlapping in the quantum case is determined by magnitude of  $\Lambda$ . This condition is reduced to that how many levels of nonlinearity of energy  $\chi$  can be located in  $U$ .

Thus, at quantum reviewing, the additional factor  $\lambda(\Lambda)$  appears in the overlapping criterion. The physical sense of  $\lambda(\Lambda)$  can be clarified. If the classic criteria of the resonances overlapping  $\sqrt{\omega'U} \approx \Omega$ , is fulfilled then, according to (50), the magnitude of  $\lambda(\Lambda)$  determines the conditions of weakening ( $\lambda(\Lambda) < 1$ ) or amplifying ( $\lambda(\Lambda) > 1$ ) the resonances overlapping criterion, in the comparison with the classical one. Differently  $\lambda(\Lambda) > 1$  means that the condition of overlapping of resonances more easily can be fulfilled in a quantum case.

The data of numerical calculations for the magnitude  $\lambda(\Lambda)$  are given on the Fig 10. As it is visible from Fig. 10,  $\lambda(\Lambda) > 1$ , i.e., the resonances overlapping criterion amplifies in the quantum case. For high values  $\Lambda$  (the quasi-classical case)  $\lambda(\Lambda)$  should tend to unit. On Fig. 11 the eigenvalues of an energy  $E/\chi$  as a functions of  $U_o$  are presented for states described by the functions  $ce_4(\alpha, U)$  and  $ce_6(\alpha, U)$ . From Figs. 10 and 11 it is easy to see that the maximum value of  $\lambda(\Lambda)$  corresponds to  $\left(\frac{\partial E}{\partial U}\right)_n = 0$ . Thus, in the quantum case the area, lying below classical separatrix, corresponds to the area of the maximum stochasticity.



## REFERENCES

- [1] R. Z. Sagdeev, D.A.Usikov, G. M. Zaslavsky, Nonlinear Physics. New-York: Hardwood Acad.Publ., 1988.
- [2] A. J. Lichtenberg, M. A. Lieberman, Regular and Stochastic Motion, Springer-Verlag, New-York, Heidelberg, Berlin, 1983.
- [3] G. M. Zaslavsky, Phys. Reports v.80,p.157,1981.
- [4] F.Haake,Quantum Signatures of Chaos,Springer,Berlin 2001.
- [5] G.P.Berman, G. M. Zaslavsky, Phys. Lett.,v.61A, p295, 1977.
- [6] G.P.Berman, G. M. Zaslavsky and A.R.Kolovsky, Phys. Lett.,v.87A, p152,1982.
- [7] M. C. Gutzwiller, Chaos in Classical and Quantum Mechanics, Springer, New York, 1990.
- [8] K. T. Alligood, T. D. Sauer and J. A. York, Chaos an introduction to Dynamical Systems Springer, New York 1996.
- [9] Handbook of Mathematical Functions Edited by M.Abramovitz, National Bureau of standards Appl.Math. series.55 , Issued June 1964.

## FIGURES

FIG. 1. Continuous line corresponds to the anharmonic potential,  $U_{NL}(x) = \frac{m\omega_o^2 x^2}{2} + \gamma x^3 + \beta x^4$ . The dashed line corresponds to the harmonic potential  $U_L(x) = \frac{m\omega_o^2 x^2}{2}$ . They coincide up to point  $x_L$

FIG. 2. Periodic series of rectangular pulses.  $\tau$  is the pulse length,  $T$  is the recurring period.

FIG. 3. Pumping spectrum  $f(t)$  consists from many harmonics, multiples to the  $\Omega = 2\pi/T$ , enveloping frequency range from  $\omega_o + 1/\tau$  up to  $\omega_o 1/\tau$ .

FIG. 4.  $\overline{\Delta I_-}$  as a function of ratio  $E/U$  in the classical case.

FIG. 5. Diffusion growth of the action obtained at the values of parameters  $f_o = 0.5, \omega_o = 20, x_l = 1, \Omega = 0.2, T = 10, \tau = 1$  and the stochasticity factor  $K = \frac{\tau}{T} f_o \omega' T \approx 3$ .

FIG. 6. Phase trajectories of the mathematical pendulum near to two resonances .

FIG. 7. Quasiclassical wave function below the separatrix.  $E = 0.9U, \Lambda \sim 100$

FIG. 8. Quasiclassical wave function near the separatrix.  $E \sim U$ .

FIG. 9. Dependence of the eigenvalue  $E$  from the  $U$  for different Mathieu's functions. Dashed line corresponds to the separatrix. Symbol  $\circ$  denotes branching points above the separatrix, symbol  $\bullet$  - branching points below the separatrix

FIG. 10. Parameter  $\lambda$  as a function of  $\Lambda$  for different Mathieu's functions  $ce_4 ce_6$ .

FIG. 11. Eigenvalue  $E$  as functions of  $U_o$  for Mathieu's functions  $ce_4 ce_6$ . Dashed line corresponds to the separatrix.





















